6. MODULES

§6.1. Modules

A (right) **R-module** is a 'vector space' M over the ring R. The 'scalars' come from R and are written on the right, so that mr ∈ M for all m ∈ M and r ∈ R. The axioms are those of a vector space, written with the scalars on the right. Note the order of the scalars in the axiom m(rs) = (mr)s. This is important if R is non-commutative.

If R has a 1 then a **unitary R-module** is one where m1 = m for all m ∈ M. **Submodules** (≤), **quotient modules** (**M/N**) and **direct sums** (**M ⊕ N**) are defined in the usual way. Vector spaces over a field F are F-modules, though in this case we usually put the scalar on the left. The fact that every finite-dimensional vector space has a basis means that it's isomorphic to the vector space

$$F \oplus F \oplus \ldots \oplus F,$$

where the number of summands is the dimension. Finite dimensional modules over a **division ring** D (a non-commutative field) also have a basis and a unique dimension, just as in standard linear algebra, and they are isomorphic to a direct sum of copies of D. (The proof is exactly as for vector spaces.)

The map θ:M→N where M, N are R-modules is an **R-module homomorphism** if it is an abelian group homomorphism and (mr)θ = (mθ)r for all m ∈ M and r ∈ R. Kernels, images and isomorphisms are defined as for vector spaces and the usual three isomorphism theorems hold.

The **annihilator** of a subset X of an R-module M is:
A(X) = $\{a \in R \mid xa = 0 \text{ for all } x \in X\}$.

Theorem 1:
(1) If $X \subseteq Y$ then $A(X) \supseteq A(Y)$.
(2) A(X) is a right ideal of R.
(3) If X is a submodule of M, A(X) is a 2-sided ideal.
Proof: (1), (2) are obvious.
(3) Let $X \leq M$, $a \in A(X)$, $r \in R$ and $m \in M$.
Then $m(ra) = (mr)a = 0$ since $mr \in X$. ✋☺

An **endomorphism** of the R-module M is a homomorphism from M to M. The set of all R-module endomorphisms of M is **End$_R$(M)**, a subring of the ring of all abelian group endomorphisms of M which is denoted by **End(M)**.

Theorem 2: If M is an R-module, R/A(M) is isomorphic to a submodule of End(M).
Proof: Let $r \in R$ and let $\Phi: M \rightarrow M$ be defined by

$$m(r\Phi) = mr.$$

Then Φ is a homomorphism with A(M) as its kernel. ✋☺

The R-module M is **faithful** if $A(M) = 0$.
Clearly M is a faithful R/A(M) module.
M is **trivial** if $A(M) = R$.

M is **irreducible** if it is non-trivial and 0 and M are its only submodules. Clearly if N is a maximal submodule of M, and M/N is non-trivial then M/N is irreducible.

Theorem 3: If M is irreducible and $0 \neq m \in M$ then

$$M = mR.$$

Proof: mR and $\{x \mid xR = 0\}$ are submodules.
Since M is non-trivial, $\{x \mid xR = 0\} = 0$.
Hence $mR = M$. ✋☺

Theorem 4 (SCHUR'S LEMMA):
If M is an irreducible R-module then $D = \mathrm{End}_R(M)$ is a division ring.
Proof: Let $0 \neq \alpha \in \mathrm{End}_R(M)$. Then $\ker \alpha = 0$ and $\mathrm{im}\, \alpha = M$ so $\alpha^{-1} \in D$. ✋☺

A ring R is **primitive** if there exists a faithful irreducible R-module. Clearly a simple ring R with a maximal right ideal I is primitive since R/I is a faithful irreducible R-module.

Example 1: $M_n(D)$, the ring of n × n matrices over the division ring D, is primitive since it is simple and

$$I = \{(a_{ij}) \mid a_{ij} = 0 \text{ if } i = 1\}$$

is a maximal right ideal.

§6.2. Annihilators of Subspaces of a Module

Theorem 5: Suppose M is an irreducible R-module and let D = $\mathrm{End}_R(M)$.
Let U be a finite dimensional subspace of M (over D).
Suppose $v \notin U$ and V = U + vD, the subspace spanned by a basis for U together with v.
Then A(V) is properly contained in A(U).

Proof: We prove the theorem by induction on $n = \dim_D U$.
Let N = {m | mR = 0}.
Then $N \leq M$ and since M is irreducible N = M or N = 0.
But if N = M then MR = 0 and so M is a trivial module, a contradiction. Hence N = 0.

If $n = 0$ then U = 0 and A(U) = R. Since $V \neq 0$, $A(V) < R$ and hence the theorem holds.

Now assume that $n > 0$ and suppose that A(V) = A(U).
For some subspace W and some $u \in U - W$, U = W + uD and, by induction, $A(W) > A(U)$. Thus uA(W) is a non-zero submodule of M and so M = uA(W).

Define $\alpha:M\to M$ by $(ua)\alpha = va$.

- **α is well-defined:** If ua = ub then $a - b \in A(U) = A(V)$ and so va = vb.

- **α ∈ D:** $(ua)r\alpha = (uar)\alpha = var = (ua)\alpha r$.

Note that $ar \in A(W)$.

- **$v - u\alpha \notin W$:** If $v - u\alpha \in W$ then $v \in W + uD$.

- **$A(W) = A(W + (v - u\alpha)D)$:**
$(v - u\alpha)a = va - u\alpha a = va - va = 0$.
This contradicts the induction hypothesis. ✋☺

Corollary: If, in the above, R has DCC on right ideals then the dimension of M over D is finite.

Proof: If $\{v_1, v_2, \ldots, v_n\}$ are linearly independent over D and $U_n = v_1D + v_2D + \ldots v_nD$ then

$$A(U_1) > A(U_2) > \ldots$$

contradicting the DCC.